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General triple charged black ring solution in supergravity

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ABSTRACT: We present the general black ring solution in $U(1)^3$ supergravity in 5 dimensions with three independent dipole and electric charges. This immediately gives the general black ring solution in the minimal 5D supergravity as well.

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1 Introduction

Black rings are a type of black hole solutions in 5-dimensional spacetime with the event horizon homeomorphic to $S^1 \times S^2 \times R$ (see [1] for a review). After the discovery of neutral black rings in pure gravity [2] the charged generalizations in supergravity were considered soon [3–6]. While the most general supersymmetric solution was found [7–10], the known families of nonextremal solutions lacked a number of free parameters. In this paper we present for the first time the general nonextremal solution in $U(1)^3$ supergravity. The solution has three independent electric and dipole charge parameters, plus 5 additional independent parameters. The general solution has the usual Dirac-Misner string and conical singularities. The condition of the absence of singularities then singles out the family of regular solutions with the complete set of independent parameters: 3 electric charges, 3 dipoles, 2 angular momenta and mass. As is well known, the corresponding solutions of the 5D minimal supergravity can be immediately obtained by setting the three electric charges and the three dipole charges equal, which makes all three gauge fields equal and all dilatons constant.

Our starting point is the solution with a single nonzero gauge field that was found in [11–13] using the inverse scattering method [14–16]. It was pointed out in [17] that when 2 gauge fields and a dilaton of $U(1)^3$ supergravity are set equal to zero, the result is the 5D Kaluza-Klein theory: the pure 6-dimensional gravity compactified on a circle. And therefore, it was possible to apply to this 6D pure gravity theory the procedure outlined in [18] based on the inverse scattering method and used in [19] to derive the general black ring solution in the 5D pure gravity. In the concluding section of [13] we suggested that the missing parameters can be introduced into the solution by symmetrizing it with respect to the dipole charges. The idea is that the general solution should be invariant under the

simultaneous permutations of the gauge fields, dilatons and corresponding dipole charges. The observed lack of symmetry is the consequence of 2 dipole charges parameters being set to some particular value and when one eliminates this asymmetry one gets the general solution. This turned out to be indeed possible to do, but first one needs to eliminate another source of asymmetry coming from a somewhat arbitrary choice of coordinates used in [13]. We explain the details of this derivation in section 3 after introducing various necessary notations in section 2. The solution is presented in section 4 using a set of functions introduced in the Appendix B. As we have checked numerically with the precision better than 10^{-100} the solution indeed satisfies the field equations written down in the Appendix A. In the concluding section 5 we discuss some possible directions for the future work.

2 Notations

Let us start by introducing the necessary notations for coordinates, parameters, fields and so on. The metric components, as well as all other fields, depend only on two coordinates: u and v . There are also three other coordinates in the 5D space-time: t, ϕ, ψ of which the metric is independent. They correspond therefore to three commuting symmetries that can be described by three commuting Killing vectors. There is also a not so short list of parameters. First of all, the four parameters $x_i, i = 0, 1, 2, 3$ give positions of poles in the inverse metric. These poles are related to the very useful notion of rods [20]. By a Möbius transformation of coordinates u and v one can take three of the x_i to the arbitrary values, while only the remaining fourth constant is indeed a parameter. A popular choice is $x_1 = -1, x_2 = 1, x_3 = -1/c, x_0 = \infty$. Then there are three parameters $a_i, i = 1, 2, 3$ that are needed to obtain by imposing the necessary regularity conditions the general nonsingular doubly rotating solution. Furthermore, there are four parameters $y_i, i = 0, 1, 2, 3$. Three of them are related to dipole charges, but y_0 is already present in the Emparan-Reall neutral black ring with a single rotation. Finally, one can apply a sequence of three boosts, interspersed with duality transformations, to charge the solution with respect to three independent electric charges. It is convenient to characterize the boosts by their velocities $\beta_i, i = 1, 2, 3$.

Despite the large number of parameters it is possible to present the solution in a relatively compact and readable form thanks to its numerous symmetries. Let us consider a set of three transformations ($i = 1, 2, 3$):

$$u \rightarrow h_i(v), v \rightarrow h_i(u), a_i \rightarrow 1/a_i. \quad (2.1)$$

Here $h_1(u)$ is the Möbius transformation with the properties: $h_1(x_1) = x_0, h_1(x_2) = x_3$ and $h_1(h_1(u)) = u$. The explicit expression is

$$h_1(u) = \frac{(x_2x_3 - x_0x_1)u + x_0x_1(x_2 + x_3) - x_2x_3(x_0 + x_1)}{(x_2 + x_3 - x_0 - x_1)u + x_0x_1 - x_2x_3}. \quad (2.2)$$

$h_2(u)$ and $h_3(u)$ can be obtained from $h_1(u)$ by exchanging x_1 with x_2 and with x_3 , respectively. All components of all fields (metric, gauge fields and scalars) of the solution do

not change under the transformations 2.1. In turn, this invariance follows from the fact that all these components have the form of ratios, and under the discussed transformation numerators and denominators of the ratios are multiplied by the same common factor. The numerators and denominators are polynomials in each a_i of degree at most 2. The invariance under 2.1 allows one to express the coefficient of a_i^2 in these polynomials in terms of the term of zero degree in a_i . To this end we introduce the symmetrization operators S_i , which act in the space of functions of u and v as follows:

$$S_i\{f(u, v)\} = f(u, v) - a_i^2 f(h_i(v), h_i(u)). \quad (2.3)$$

The operators S_i commute with each other. The composition of all S_i will be denoted by $S\{f\} = S_1\{S_2\{S_3\{f\}\}\}$.

We never use summation over repeating indices, except when the sum is written explicitly. We use the convention that the indices i, j, k, l have arbitrary but different values, in other words they represent a permutation of $(0, 1, 2, 3)$. We will use frequently 6-component quantities, where each component corresponds to an unordered pair of indices i and j ($i \neq j$). The addition and multiplication for them is the component-wise one. The double vertical line brackets $||...||$ will denote the sum of all 6 components of the quantity in the brackets. We introduce two functions of a single variable $l(z)$ and $q(z)$ that take such 6-component values:

$$l_{ij}(z) = (z - x_i)(z - x_j), \quad q_{ij}(z) = \sqrt{z - x_i} \sqrt{z - x_j}, \quad (2.4)$$

and a 6-component valued function of two variables $r(u, v)$:

$$r_{ij}(u, v) = \frac{(u - x_i)(v - x_i)}{G'(x_i)} + \frac{(u - x_j)(v - x_j)}{G'(x_j)}, \quad (2.5)$$

where

$$G(u) = \prod_{i=0}^3 (u - x_i). \quad (2.6)$$

Let us introduce also a constant 6-component quantity Δ with the components equal to $\Delta_{ij} = (x_i - x_j)^2$. For a 6-component quantity we will denote by bar the transposition, which consists in exchanging ij with kl components, for example: $\bar{q}_{ij} = q_{kl}$. Note that $r_{ij}(u, v) = -r_{kl}(u, v)$, or with the above notation: $\bar{r}(u, v) = -r(u, v)$. It is also useful to combine the constants a_i into the 6-component constant a with the components $a_{0i} = a_i$, and the property $\bar{a} = -a$. Let us introduce also a trilinear function of 3 variables $c(p, s, t)$, where each of the variables s, p and t is a 6-component quantity, and the function is equal to

$$c(p, s, t) = \frac{1}{2} \sum_{i \neq j \neq k} p_{ij} s_{jk} t_{ki}, \quad (2.7)$$

where the sum is over all 24 ordered triplets (i, j, k) of non-equal values of indices. Such function corresponds to a totally symmetric tensor of rank 3. When one takes all three arguments equal, one obtains a cubic function that we will call c_3 : $c_3(t) = \frac{1}{3}c(t, t, t)$. We

will use also a symmetrization operator S' , which acts on the 6-component quantities as follows:

$$S'(t)_{0i} = S_j(S_k(t_{0i})), \quad S'(t)_{ij} = S_i(S_j(t_{ij})), \quad (2.8)$$

where the triple $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$.

Having introduced all these notations, we are now ready to write down the complete set of functions of coordinates and parameters which appear in the solution. This is done in the Appendix B. This set is comprised of ten functions H_{ij} , K_{ij} , Ω^{ij} , ω_i , Σ_i , Π_{ij} , Ξ_{ij} , Q , Z , and $g^{\varphi\psi}$.

3 Derivation

Let us now explain how we were able to recover the general family of solutions which is the subject of this paper, starting from its particular subfamily presented in our previous paper [13]. The solution in [13] depends among others on 2 parameters y_1 and y_2 . It will be convenient to shift the index and rename the parameters y_0 and y_1 . Recall also, that in [13] we had the following choice of parameters that specify the positions of poles in the inverse metric: x_i for $i = 1, 2, 3$ in arbitrary positions, and the fourth pole fixed at the infinity: $x_0 = \infty$. If one wants to put the solution in the maximally symmetric form, one would like to treat all x_i on equal footing and make x_0 a free parameter too. This can be done by a coordinate transformation, making substitutions $u \rightarrow h(u)$ and $v \rightarrow h(v)$, where $h(v)$ is a Möbius transformation that has a pole at the point x_0 :

$$h(u) = \frac{\alpha u + \beta}{u - x_0}. \quad (3.1)$$

One has to make also the same substitution for the constants: $x_i \rightarrow h(x_i)$, $i = 1, 2, 3$ and $y_i \rightarrow h(y_i)$. At the same time one rescales the constants

$$a_i \rightarrow a_i \frac{\beta + \alpha x_0}{x_i - x_0} \frac{\sqrt{x_i - y_0} \sqrt{x_i - y_1}}{\sqrt{x_0 - y_0} \sqrt{x_0 - y_1}}, \quad (3.2)$$

$i = 1, 2, 3$ in order to eliminate everywhere the dependence on α and β and to put x_0 on equal footing with other x_i .

In the new coordinates the solution has the following instructive property. Let us take $g^{\varphi\varphi}$ component as the simplest example. It contains a term with the following product $(u - x_0)^2(u - y_0)(u - y_1)$. One can say that the explicit dependence on x_0 , which violates the symmetry among all x_i , is the effect of setting $y_2 = y_3 = x_0$ in the symmetric general solution. We see that if one replaces $(u - x_0)^2$ by $(u - y_2)(u - y_3)$, one restores both symmetry and generality (hopefully) at the same time. We have applied this idea systematically to all components of inverse metric, gauge fields and scalars. A minor complication is that not all components should be symmetric in all y_i . Only $g^{\varphi\varphi}$, $g^{\psi\psi}$ and $g^{\varphi\psi}$ have this total symmetry. Other components of the inverse metric, namely g^{tt} , $g^{t\varphi}$ and $g^{t\psi}$, single out y_0 but should be symmetric in y_1 , y_2 and y_3 . The gauge fields A_j single out the corresponding y_j and y_0 , but should be symmetric in the remaining two y_i ($i \neq j \neq 0$). The same is true for scalar fields. After one gains some experience, the symmetrization procedure becomes almost

straightforward, only with a small amount of guesswork needed. Fortunately, there is an excellent way to check the correctness for each component of the inverse metric separately: the residues at the poles in u should not depend on v apart from a common factor which is easy to cancel completely. Furthermore, the residues should factorize: $\text{res}(g^{ij}) \sim \rho^i \rho^j$, where ρ^i is a constant vector – the rod direction (for the description of rods see [20]). This test is very strict because it is extremely improbable to have such factorization to hold by chance in an incorrect expression. When all components of the inverse metric have been obtained, one more test becomes available: it turns out that the determinant of the 3×3 matrix of inverse metric components for coordinates t , φ and ψ has a very simple form $\det(g^{ij}) = \frac{(u-v)^4}{G(u)G(v)}$. After all fields have been found, one can finally check that they indeed satisfy the field equations. Due to the high enough complexity of the solution, we were not able to perform this check analytically even with the help of a computer algebra system. Instead, it was possible to do this numerically with a precision of more than 100 digits. Such numerical precision is absolutely sufficient to convince everyone that the solution is correct. We used Wolfram Mathematica for both algebraic manipulations and numerical calculations. A Mathematica notebook that contains the solution and the numerical checks of the field equations is available on request.

Once the solution with general values of the dipole charges is found one can turn on the electric charges too. There is a well-known procedure for charging a 5D $U(1)^3$ supergravity solution (see e.g. [6, 21–23]). It can be done by uplifting the solution to six dimensions, treating one of the gauge fields as a Kaluza-Klein one, and making a boost along the compact sixth direction. The other two gauge fields at the same time combine into the 2-form. Then one can reduce the result back to five dimensions and repeat the procedure with the next gauge field. After three boosts one obtains the general solution with three independent electric charges. The charges are parametrized by the velocities β_i of the boosts.

4 Solution

In this section we will present the general black ring solution of the 5D $U(1)^3$ supergravity field equations. The field equations themselves are written down in the Appendix A. We will express the components of the fields in terms of a set of auxiliary functions defined in the Appendix B. Let us start from the scalar fields Φ_i . They have the following form:

$$e^{\Phi_i} = \frac{\chi_i}{\chi}, \quad \chi_i = H_{0i} - \beta_i^2 \tilde{H}_{0i} + 2\beta_i K_{0i}, \quad \chi = \prod_{i=1}^3 \chi_i^{1/3}. \quad (4.1)$$

The tt component of the inverse metric is:

$$g^{tt} = \chi^{-1} \left(\sum_{m,n=0}^3 \left(-2\beta_0 \frac{\beta_n}{\beta_m} \Pi_{mn} - \beta_0^2 \frac{\Xi_{mn}}{\beta_m \beta_n} \right) + \sum_{m=0}^3 \left(\beta_0^2 \frac{\Sigma_m}{\beta_m^2} - \beta_m^2 \tilde{\Sigma}_m \right) + 2\beta_0 Q \right),$$

where the diagonal elements of Π and Ξ are defined to be zero: $\Pi_{mm} = \Xi_{mm} = 0$ and a shorthand notation $\beta_0 = -\beta_1 \beta_2 \beta_3$ is introduced. The other non-zero inverse metric

components are:

$$g^{t\varphi} = \chi^{-1} \sum_{m=0}^3 \left(\beta_m \tilde{\omega}_m^\psi + \frac{\beta_0}{\beta_m} \omega_m^\varphi \right), \quad g^{\varphi\varphi} = \frac{Z}{\chi}, \quad g^{\psi\psi} = -\frac{\tilde{Z}}{\chi}, \quad (4.2)$$

$$g^{\varphi\psi} = -\chi^{-1} \left\| a \frac{r(u,v)}{u-v} q(y_0)q(y_1)q(y_2)q(y_3) S' \left\{ \frac{(u-v)^2}{l(u)l(v)} \right\} \right\| \\ + \frac{u-v}{\chi G(u)G(v)} \prod_{m=0}^3 G(y_m) c_3 \left(\frac{a r(u,v) \Delta}{q(y_0)q(y_1)q(y_2)q(y_3)} \right). \quad (4.3)$$

$$g^{uu} = \frac{(u-v)^2}{C_0 \chi} G(u), \quad g^{vv} = -\frac{(u-v)^2}{C_0 \chi} G(v), \quad (4.4)$$

where C_0 is an arbitrary constant.

The determinant of the 3×3 matrix of the inverse metric components g^{mn} ($m, n = t, \varphi, \psi$) has the simple form:

$$\det(g^{mn}) = \frac{(u-v)^4}{G(u)G(v)}. \quad (4.5)$$

The gauge vector potentials are:

$$A_t^i = \frac{1}{\chi_i} \left(\frac{1 + \beta_i^2}{1 - \beta_i^2} K_{0i} + \frac{\beta_i}{1 - \beta_i^2} (H_{0i} - \tilde{H}_{0i}) \right), \\ A_\varphi^i = \frac{1}{\chi_i} \left(\Omega_\varphi^{i0} - \beta_j \beta_k \Omega_\varphi^{0i} + \beta_i \beta_j \Omega_\varphi^{jk} + \beta_i \beta_k \Omega_\varphi^{kj} + \beta_1 \beta_2 \beta_3 \tilde{\Omega}_\psi^{i0} - \beta_i \tilde{\Omega}_\psi^{0i} + \beta_k \tilde{\Omega}_\psi^{jk} + \beta_j \tilde{\Omega}_\psi^{kj} \right).$$

A_ψ^i can be obtained from A_φ^i by exchanging all ϕ and ψ subscripts in the expression above.

5 Conclusions

In this paper we presented the general black ring solution in $U(1)^3$ supergravity (and therefore in the minimal 5D supergravity as well). We have tried to simplify it as much as possible. To this end we introduced several notations, which allowed to reduce the length of expressions considerably. Still we are not completely satisfied at this point with the form of the solution. One can hope that there is a formulation that is both elegant and allows to check the validity of the solution analytically, instead of checking it numerically, as we were forced to do. Such formulation would uncover the natural algebraic structure of the solution and give it a nice mathematical sense.

One possible way to reach such better understanding of the solution is to try to generalize it to a larger supergravity that reduces to the $U(1)^3$ theory when some fields vanish. Interesting examples are 11D supergravity (the low energy limit of M-theory) reduced to 5D on T^6 or on $K3 \times T^2$ which gives theories with 27 gauge fields [24, 25]. In the first case some of the missing dipole charges can be generated by suitably uplifting our solution to 11D and then rotating it in the six compact directions. Combining this rotations with duality transformations can probably generate even more independent dipole charges.

As an intermediate step one could also try to find the black ring solutions in the theory with just one additional gauge field considered in [23]. It would be straightforward to add an electric charge with respect to the additional gauge field. Adding the dipole charge using symmetry considerations is not straightforward, but may turn out to be possible with some luck. Another interesting problem is a deeper investigation of the black ring solution in minimal 5D supergravity. In this case the number of independent parameters and fields is smaller than in more general case of $U(1)^3$ supergravity, and therefore one can hope to be able to get more explicit expressions for the regular solution and its mass, angular momenta etc.

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A Field equations

In this appendix the Greek indices enumerate the five-dimensional spacetime coordinates, and the summation over repeating Greek indices is assumed. The field equations for the $U(1)^3$ five-dimensional supergravity can be derived from the following action:

$$I = \int d^5x \sqrt{-g} \left(R - \frac{1}{4} \sum_{i=1}^3 e^{2\Phi_i} F_{\mu\nu}^i F^{i\mu\nu} - \frac{1}{2} \sum_{i=1}^3 g^{\mu\nu} \partial_\mu \Phi_i \partial_\nu \Phi_i \right) - \int dA^1 \wedge dA^2 \wedge dA^3, \quad (\text{A.1})$$

with the constraint $\Phi_1 + \Phi_2 + \Phi_3 = 0$ and where $F^i = dA^i$. The resulting field equations have the form

$$\begin{aligned}\partial_\nu (\sqrt{-g} e^{a\Phi_i} F^{i\sigma\nu}) &= \frac{1}{4} \epsilon^{\mu\nu\kappa\lambda\sigma} F_{\mu\nu}^j F_{\kappa\lambda}^k, \\ \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi_i) &= \frac{\sqrt{-g}}{6} \left(2e^{2\Phi_i} F_{\mu\nu}^i F^{i\mu\nu} - e^{2\Phi_j} F_{\mu\nu}^j F^{j\mu\nu} - e^{2\Phi_k} F_{\mu\nu}^k F^{k\mu\nu} \right), \\ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \frac{1}{2} \sum_{i=1}^3 e^{2\Phi_i} \left(F_{\lambda\mu}^i F^{i\lambda}{}_\nu - \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda}^i F^{i\kappa\lambda} \right) \\ &\quad + \frac{1}{2} \sum_{i=1}^3 \left(\partial_\mu \Phi_i \partial_\nu \Phi_i - \frac{1}{2} g_{\mu\nu} g^{\kappa\lambda} \partial_\kappa \Phi_i \partial_\lambda \Phi_i \right).\end{aligned}\tag{A.2}$$

Let us write down the equations for the gauge field strength more explicitly:

$$\begin{aligned}\partial_u \left(\frac{G(u)}{(u-v)^2} e^{2\Phi_i} \sum_{n=t,\varphi,\psi} g^{mn} \partial_u A_n^i - \sum_{n=t,\varphi,\psi} \sum_{s=t,\varphi,\psi} \epsilon^{mns} A_n^j \partial_v A_s^k \right) \\ = \partial_v \left(\frac{G(v)}{(u-v)^2} e^{2\Phi_i} \sum_{n=t,\varphi,\psi} g^{mn} \partial_v A_n^i - \sum_{n=t,\varphi,\psi} \sum_{s=t,\varphi,\psi} \epsilon^{mns} A_n^j \partial_u A_s^k \right),\end{aligned}\tag{A.3}$$

where ϵ^{mns} is the antisymmetric tensor and $\epsilon^{t\varphi\psi} = 1$. The field equations for the dilatons can be reduced to the statement, that the expression

$$\begin{aligned}\partial_u \left(\frac{G(u)}{(u-v)^2} \partial_u \Phi_i \right) - \partial_v \left(\frac{G(v)}{(u-v)^2} \partial_v \Phi_i \right) \\ - \sum_{m,n=t,\varphi,\psi} e^{2\Phi_i} \left(\frac{G(u)}{(u-v)^2} g^{mn} \partial_u A_m^i \partial_u A_n^i - \frac{G(v)}{(u-v)^2} g^{mn} \partial_v A_m^i \partial_v A_n^i \right)\end{aligned}\tag{A.4}$$

does not depend on i and the constraint $\Phi_1 + \Phi_2 + \Phi_3 = 0$ is satisfied. Finally, the Einstein equations can be reduced to the following form:

$$\begin{aligned}\partial_u \left(\frac{G(u)}{(u-v)^2} \sum_{s=t,\varphi,\psi} g_{ms} \partial_u g^{sn} \right) - \partial_v \left(\frac{G(v)}{(u-v)^2} \sum_{s=t,\varphi,\psi} g_{ms} \partial_v g^{sn} \right) \\ = \sum_{i=1}^3 \sum_{s=t,\varphi,\psi} e^{2\Phi_i} \left(\frac{G(u)}{(u-v)^2} \partial_u A_m^i g^{ns} \partial_u A_s^i - \frac{G(v)}{(u-v)^2} \partial_v A_m^i g^{ns} \partial_v A_s^i \right) \\ - \frac{1}{3} \delta_m^n \sum_{i=1}^3 \sum_{p,s=t,\varphi,\psi} e^{2\Phi_i} \left(\frac{G(u)}{(u-v)^2} \partial_u A_p^i g^{ps} \partial_u A_s^i - \frac{G(v)}{(u-v)^2} \partial_v A_p^i g^{ps} \partial_v A_s^i \right)\end{aligned}\tag{A.5}$$

B Set of Functions

In this appendix we define the set of function used in section 4 to write down the black ring solution. The notations used here were described in section 2.

$$\begin{aligned}H_{ij} &= -S \left\{ \frac{1}{(u-v)^2} (u-y_i)(u-y_j)(v-y_k)(v-y_l) \right\} \\ &\quad + \frac{1}{(u-v)^2} c(a r(u, v), a r(u, v), \Delta q(y_i) q(y_j) q(y_k) q(y_l)).\end{aligned}\tag{B.1}$$

$$K_{ij} = \|S' \left\{ \frac{a r(u, v)}{u - v} \right\} q(y_i) q(y_j) \bar{q}(y_k) \bar{q}(y_l)\|. \quad (\text{B.2})$$

$$\begin{aligned} \Omega_\varphi^{ij} &= \left\| \frac{a r(u, v)}{u - v} q(y_i) \bar{q}(y_j) \bar{q}(y_k) \bar{q}(y_l) S' \left\{ u - y_j - \frac{l(u)}{u - v} \right\} \right\|, \\ \Omega_\psi^{ij} &= - \prod_{m=1}^4 \sqrt{x_m - y_i} \frac{\partial H_{ij}}{\partial y_i}. \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} Z &= -S \left\{ \frac{1}{G(v)} (v - y_0)(v - y_1)(v - y_2)(v - y_3) \right\} \\ &+ c \left(a r(u, v), a r(u, v), \bar{\Delta} q(y_0) q(y_1) q(y_2) q(y_3) \left(\frac{1}{\bar{l}(u) l(v)} - \frac{a^2}{l(u) \bar{l}(v)} \right) \right). \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \omega_i^\varphi &= \prod_{m=1}^4 \sqrt{x_m - y_i} \frac{\partial Z}{\partial y_i}, \\ \omega_i^\psi &= \left\| \frac{a r(u, v)}{u - v} \bar{q}(y_i) q(y_j) q(y_k) q(y_l) S' \left\{ \frac{u - v}{l(v)} \left(1 - \frac{u - v}{l(u)} (u - y_i) \right) \right\} \right\| \\ &- \frac{u - v}{G(u) G(v)} \prod_{m=1}^4 \sqrt{x_m - y_i} G(y_j) G(y_k) G(y_l) c_3 \left(\frac{a r(u, v) \Delta}{q(y_j) q(y_k) q(y_l)} \right). \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \Sigma_i &= S \left\{ \left(\frac{(u - y_i)^2}{(u - v)^2} - \frac{G(y_i)}{G(v)} \right) \frac{(v - y_j)(v - y_k)(v - y_l)}{(v - y_i)} \right\} \\ &- c \left(a r(u, v), a r(u, v), \Delta q(y_1) q(y_2) q(y_3) q(y_4) \left(\frac{1 - a^2}{(u - v)^2} - \bar{l}(y_i) \left(\frac{1}{\bar{l}(u) l(v)} - \frac{a^2}{l(u) \bar{l}(v)} \right) \right) \right). \end{aligned}$$

$$\begin{aligned} \Pi_{ij} &= \left\| \frac{a r(u, v)}{u - v} \bar{q}(y_i) \bar{q}(y_j) q(y_k) q(y_l) S' \left\{ \left(1 - \frac{u - v}{l(u)} (u - y_i) \right) \left(1 - \frac{v - u}{l(v)} (v - y_j) \right) \right\} \right\| \\ &+ \frac{u - v}{G(u) G(v)} G(y_k) G(y_l) \prod_{m=1}^4 \sqrt{x_m - y_i} \sqrt{x_m - y_j} c_3 \left(\frac{a r(u, v) \Delta}{q(y_k) q(y_l)} \right). \end{aligned}$$

$$\begin{aligned} \Xi_{ij} &= - \prod_{m=1}^4 \sqrt{x_m - y_i} \sqrt{x_m - y_j} \frac{\partial^2 Z}{\partial y_i \partial y_j} - \prod_{m=1}^4 \sqrt{x_m - y_k} \sqrt{x_m - y_l} \frac{\partial^2 Z}{\partial y_k \partial y_l} \\ &+ \frac{1}{2} c \left(a r(u, v), a r(u, v), \Delta \bar{\Delta} q(y_i) q(y_j) \bar{q}(y_k) \bar{q}(y_l) \left(\frac{1}{\bar{l}(u) l(v)} - \frac{a^2}{l(u) \bar{l}(v)} \right) \right). \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} Q &= \left\| a \frac{r(u, v)}{u - v} q(y_0) q(y_1) q(y_2) q(y_3) (\bar{l}(y_0) + \bar{l}(y_1) + \bar{l}(y_2) + \bar{l}(y_3)) S' \left\{ \frac{(u - v)^2}{l(u) l(v)} \right\} \right\| \\ &- \frac{u - v}{G(u) G(v)} \prod_{m=0}^3 G(y_m) \sum_{n=0}^3 \frac{1}{G(y_n)} c_3 \left(\frac{a r(u, v) q(y_n) \Delta}{q(y_0) q(y_1) q(y_2) q(y_3)} \right). \end{aligned}$$